Quantum Measurements

(recap)

Measurements of classical systems

M_1: "What colour is S?" Three possible outcomes 'red', 'green' or 'blue' Label these answers 0, 1, 2.

$$\boxed{\begin{aligned} \text{Prob}[\textbf{red}] &= \text{Prob}[0] = (1,0,0)\cdot(r_0,r_1,r_2) = r_0 \\ \text{Prob}[\textbf{green}] &= \text{Prob}[1] = (0,1,0)\cdot(r_0,r_1,r_2) = r_1 \\ \text{Prob}[\textbf{blue}] &= \text{Prob}[2] = (0,0,1)\cdot(r_0,r_1,r_2) = r_2. \end{aligned}}$$

$M_1 = (m_0, m_1, m_2)$

$m_0 = (1, 0, 0)$
$m_1 = (0, 1, 0)$
$m_2 = (0, 0, 1)$

M_2: "Is S blue?" Two possible outcomes 'yes' or 'no'. Label these answers 0 and 1.

$$\boxed{\begin{aligned} \text{Prob}[\text{‘}no\text{’}] &= \text{Prob}[0] = (1,1,0)\cdot(r_0,r_1,r_2) = r_0 + r_1 \\ \text{Prob}[\text{‘}yes\text{’}] &= \text{Prob}[1] = (0,0,1)\cdot(r_0,r_1,r_2) = r_2. \end{aligned}}$$

$M_2 = (m_0, m_1)$

$m_0 = (1, 1, 0)$
$m_1 = (0, 0, 1)$

General rule $\boxed{\text{Prob}[k] = \boldsymbol{m}_k \cdot \boldsymbol{r}}$

- Components of each $\boldsymbol{m}_k$ must be positive
- And need $\sum_k \boldsymbol{m}_k = (1, 1, \dots, 1)$

Measurements of quantum systems

Given a classical system S with a total of n pure states. Any measurement $\mathcal{M}$ with d outcomes is described by a collection of vectors $\{\boldsymbol{m}_0, \boldsymbol{m}_1, \dots, \boldsymbol{m}_{d-1}\}$, such that

- (Non-negativity) For any k the components of $\boldsymbol{m}_k$ are non-negative.
- (Normalisation) $\sum_k \boldsymbol{m}_k = (1, 1, 1, \dots, 1)$.

and the probability of getting outcome k in the state $\boldsymbol{r}$ is given by

$$\text{Prob}[k] = \boldsymbol{m}_k \cdot \boldsymbol{r}, \tag{13}$$

and so the vectors $\{\boldsymbol{m}_k\}$ are of length n.

vector goes to matrix & dot product goes to matrix inner product

Given a quantum system S of dimension d, any general quantum measurement $\mathcal{M}$ on S with m outcomes is described by a set of matrices $\mathcal{M} = \{M_0, M_1, \dots, M_{m-1}\}$ such that

1. (Non-negativity) $\underline{eigs(M_i) \geq 0}$ for all $i = 0, 1, \dots m-1$.
2. (Normalization) $\sum_i M_i = \mathbb{1}$.

Where $\mathbb{1}$ is the $d \times d$ identity matrix. Moreover, given a quantum state ρ, the probability of the k'th outcome of doing the measurement $\mathcal{M}$ on the state is given by

$$\text{Prob}[k] = \text{tr}[M_k \rho]. \tag{16}$$

POVM Measurements : Positive Operator Value Measurements

- doesn't say anything about how the measurement is realised
- doesn't say anything about post measurement state

Examples: 1 Projective Measurements (how POVMs & Observables are related)

$\mathcal{M} = \{ M_0 = |+\rangle\langle+| , \; M_1 = |-\rangle\langle-| \}$

Positivity ✓ (both M_0 & M_1 have eigenvalues 0 & 1)

Normalization ✓ $\quad M_0 + M_1 = |+\rangle\langle+| + |-\rangle\langle-| = \mathbb{I}$

$\therefore$ Yes, this is an example of a POVM measurement

Projective measurements satisfy (by definition) a third property

Orthogonality $\quad M_i M_j = \delta_{ij} M_i$

It follows that there are always d projectors in $\mathcal{M}$ for a projective measurement on a d dimension system.

In this case, if the answer corresponding to measuring $\underbrace{M_i = \Pi_i}_{\text{where } \Pi_i \Pi_j = \delta_{ij}}$ is associated with a numerical value λ_i then we can associate the measurement with a Hermitian operator

$$O = \sum_i \lambda_i \Pi_i$$

which in the standard story is written as $|e_i\rangle\langle e_i|$

& thus we regain the standard story about Measurement in Intro to Quantum text books.

In this case (as you all know) the average output of the measurement is $\quad \langle O \rangle_\rho = \mathrm{Tr}(\rho O)$

& the post measurement state on obtaining λ_i is standardly taken to be $\Pi_i \rho \Pi_i$

But note, to regain this story we [illegible] our model, [illegible] we make two additional assumptions

1. Associating the lable of the measurement with a numerical value
2. Associating the output lable with a specific quantum state & a rule as to how this state is obtained from measurements.
   This rule is chosen such that repeated measurements of a property known with certainty give the same measurement outcome.

These extra assumptions are natural but not essential & rather limiting.

(2) Convex mixtures of projectors

$$M = \left\{ \tfrac{1}{2}|0\rangle\langle 0|,\ \tfrac{1}{2}|1\rangle\langle 1|,\ \tfrac{1}{2}|+\rangle\langle +|,\ \tfrac{1}{2}|-\rangle\langle -| \right\}$$

Positivity ✓ (all have eigenvalues 0, 1)

Normalisation ✓ $\quad \underbrace{\tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|}_{I/2} + \underbrace{\tfrac{1}{2}|+\rangle\langle +| + \tfrac{1}{2}|-\rangle\langle -|}_{I/2} = I$

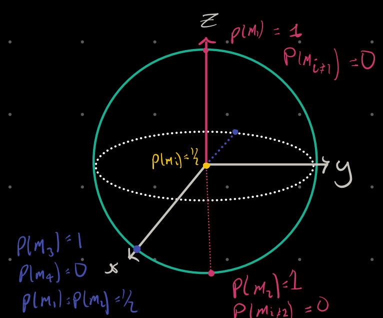
Interpretation?

With probability 1/2, measure X
" 1/2, measure Z

you could if you were being annoying question whether this really counts as a single measurement

Why would anyone do this? Well it allows you (with a 'single' measurement) to get information about both the X & Z components of a state

Can't get this information with a single text book projective measurement



③ State Discrimination (N&C pg. 90)

Suppose Alice gives Bob a qubit prepared in either

$$\underbrace{|0\rangle}_{|\psi_1\rangle} \quad \text{or} \quad \underbrace{|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|\psi_2\rangle}$$

It is impossible for Bob to determine whether he can distinguish $|0\rangle$ or $|+\rangle$ with certainty

(More on this in a later wk but for now note that measuring in either the Z or X basis clearly doesn't work)

Using POVM's it's possible to perform a measurement which distinguishes the states with certainty some of the time but gives a completely inconclusive answer other times

$$M = \{ M_1, M_2, M_3 \}$$

$$M_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1|$$
$$M_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |-\rangle\langle -|$$

} positive ✓

$$M_3 = I - M_1 - M_2$$

normalisation ✓ positive ✓ (eigenvalues 0 & 1) (check this!)

How does it work?

Well, if you measure M_1, you know for certainty you don't have $|\psi_1\rangle$, so you know you have $|\psi_2\rangle$

Similarly if you measure M_1, you know you have $|\psi_2\rangle$

But if you measure M_3 the result is inconclusive

} Never misidentifies state.

4) Non-rank 1 POVM measurements

Previous examples all involve rank 1 measurements where the each M_i has at most 1 non-zero eigenvalue.

Can also have cases where the no. of non-zero eigenvalues is greater than 1.

eg. $\mathcal{M} = \{ M_0, M_1 \}$ ← 2 outcome measurement on 2-qubit $(d=4)$ system

$M_0 = |\psi^-\rangle\langle\psi^-|$

$M_1 = |\phi^+\rangle\langle\phi^+| + |\phi^-\rangle\langle\phi^-| + |\psi^-\rangle\langle\psi^-|$

This measurement addresses the question:

"Is the system in the singlet state?"

$M_0 \equiv$ "Yes"

$M_1 \equiv$ "No"

M_0 has eigenvalues 0, 1 as previously

$M_1 = |00\rangle\langle 00| + |\psi^-\rangle\langle\psi^-| + |11\rangle\langle 11|$

has eigenvalues $(1, 1, 1, 0)$

corresponding to the eigenvectors

Note - these examples highlight that the number of measurements need not coincide with the dimension of the system (contrary to the standard story of projective measurements).

This is entirely natural there are many different questions you can ask about a system with different numbers of outcomes. These questions can be realised by POVM measurements beyond standard projective measurements.

That said, similarly to purifications, any POVM measurement can actually be realised as a projective measurement on a larger system ....

Naimark's Dilation Theorem

If $\{M_i\}_{i=1}^{n}$ is a POVM acting on a Hilbert space $\mathcal{H}_A$ of dimension d_A, then there exists a projective measurement $\{\Pi_i\}_{i=1}^{n}$ acting on a Hilbert space of dimension $d_{A'}$ and an isometry $V: \mathcal{H}_A \to \mathcal{H}_{A'}$ such that $\forall i$

$$M_i = V^\dagger \Pi_i V$$

An isometry is just a transformation from a given Hilbert space into a potentially larger Hilbert space.

$$U^\dagger U = I \qquad (\text{but } UU^\dagger \text{ does not necessarily} = I)$$

They can be formed by taking a unitary on the larger space & deleting some columns.

Say $U = \begin{pmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ & & \end{pmatrix}$

$$U^\dagger U = \begin{pmatrix} \langle u_1|u_1\rangle & \langle u_1|u_2\rangle & \cdots \\ \langle u_1|u_1\rangle & \ddots & \\ \vdots & & \end{pmatrix}$$

$$U^\dagger = \begin{pmatrix} \underline{u}_1^* \\ \underline{u}_2^* \\ \underline{u}_3^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$UU^\dagger = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} \underline{u}_1 & \underline{u}_2 \\ & \end{pmatrix}$$

$$V^\dagger V = \begin{pmatrix} \underline{u}_1^* \\ \underline{u}_2^* \end{pmatrix} \begin{pmatrix} \underline{u}_1 & \underline{u}_2 \\ & \end{pmatrix}$$

$$= \begin{pmatrix} \langle u_1|u_1\rangle & \langle u_1|u_2\rangle \\ \langle u_1|u_1\rangle & \langle u_2|u_2\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$VV^\dagger \neq I$$

One possible construction of projective measurement that satisfies Naimark's Dilation Theorem is the following.

$$\Pi_i = I_A \otimes |i\rangle\langle i|_B$$

$$V = \sum_{i=1}^{n} \sqrt{M_i}_A \otimes |i\rangle_B$$

dimension of B needs to be same as no. of measurement outcomes here.
(more efficient constructions can also be possible)

The probability of measuring Π_i on $V\rho_A V^\dagger$ is the same as measuring M_i on ρ_A.

$$\mathrm{Tr}(V \rho V^\dagger \Pi_i) = \mathrm{Tr}(V \rho_A V^\dagger (I_A \otimes |i\rangle\langle i|_B))$$

$$= \sum_{jk} \mathrm{Tr}\left(\rho_A \left(\sqrt{M_j}_A \otimes \langle j|_B\right)\left(I_A \otimes |i\rangle\langle i|_B\right)\left(\sqrt{M_k}_A \otimes |k\rangle_B\right)\right)$$

$$\delta_{ji} \qquad \delta_{ik}$$

$$= \mathrm{Tr}_A(\rho_A M_i)$$

This construction is always possible & hence Naimark's theorem holds as claimed.